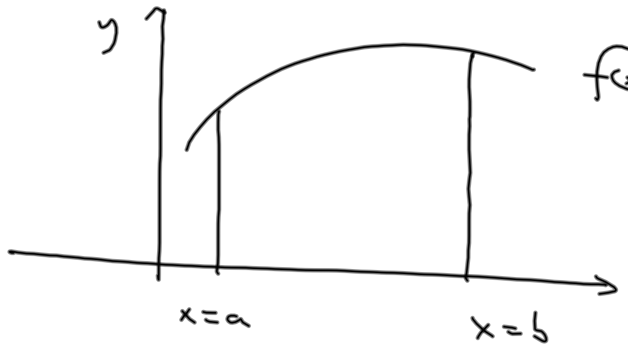
Numerical Techniques:

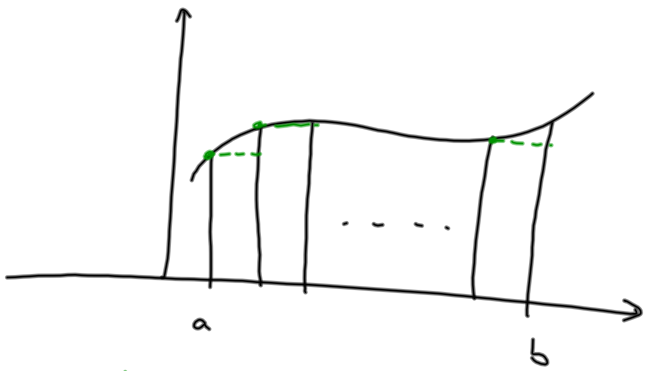


$$\text{Area} = \int_a^b f(x)\,dx$$
$$= F(b) - F(a)$$

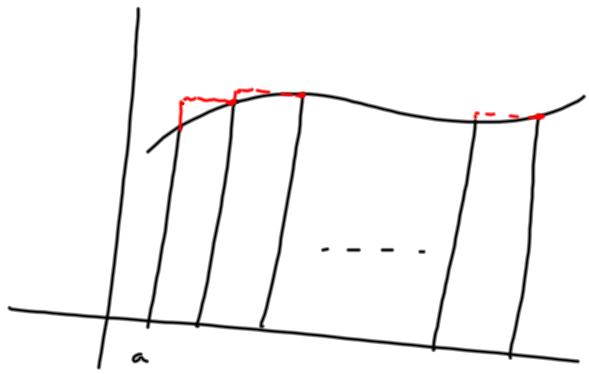
What if we cannot find $F(x)$, ie, a function which derivative is $f(x)$

We rely (in general) on a numerical approximation technique. Here we'll look at:

- Riemman's Sum (ok)
- Trapezoidal Rule (good)
- Simpson's Rule (better)



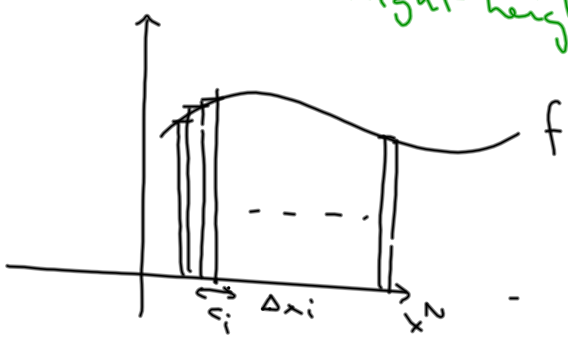
left heights



Right heights

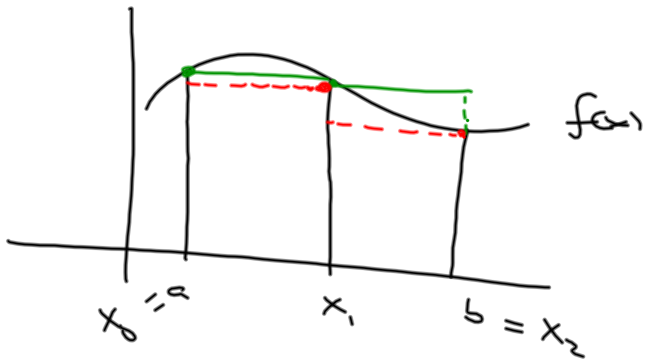
$$\lim_{\|P\| \to 0} \sum_{i=1}^{N} f(c_i)\,\Delta x_i$$

$$= \int_a^b f(x)\,dx$$



N- rectangles of base $\|P\|$

Riemman's Sum:



Region subdivide into $n=2$ rectangles of equal base: $\frac{b-a}{n}$

heights: Left: $f(x_0)$, $f(x_1)$ or $\frac{b-a}{2}$ (when $n=2$)

Right: $f(x_1)$, $f(x_2)$

Area: (using left heights) $\longrightarrow$ Left Riemman's Sum: L_2

= base . height

$$\text{Area} = \frac{b-a}{2} f(x_0) + \frac{b-a}{2} f(x_1)$$

$$= \frac{b-a}{2}\left[f(x_0) + f(x_1) \right]$$

Likewise, using the right heights, $\rightarrow$ Right Riemman's Sum: R_2

$$\text{Area} = \frac{b-a}{2}\left[f(x_1) + f(x_2) \right]$$

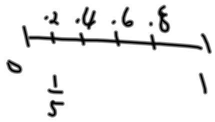
In general, if the region over $[a,b]$ of $f(x)$ is subdivided into n rectangles, each has base $\frac{b-a}{n}$, and the Riemman's Sums are:

Left: $L_n = \frac{b-a}{n}\left[f(x_0)+f(x_1)+\dots+f(x_{n-1})\right]$

Right: $R_n : \frac{b-a}{n}\left[f(x_1)+f(x_2)+\dots+f(x_n)\right]$

#11 Compute R_5 and L_5 over $[0,1]$ using the following

x	0	0.2	0.4	0.6	0.8	1
f(x)	50	47	46	45	44	41



$\frac{b-a}{n}$ = base: .2

$n = 5$

$a=0, b=1$

So $R_5 = .2\left[47+46+45+44+41\right]$

$\boxed{R_5 \simeq 44.6}$

$L_5 = .2\left[50+47+46+45+44\right]$

$\boxed{L_5 \simeq 46.4}$

#12 $f(x) = 5x+2$ on $[0,3]$

Compute $\underline{\underline{L_6}}$ and R_6 over $[0,3]$

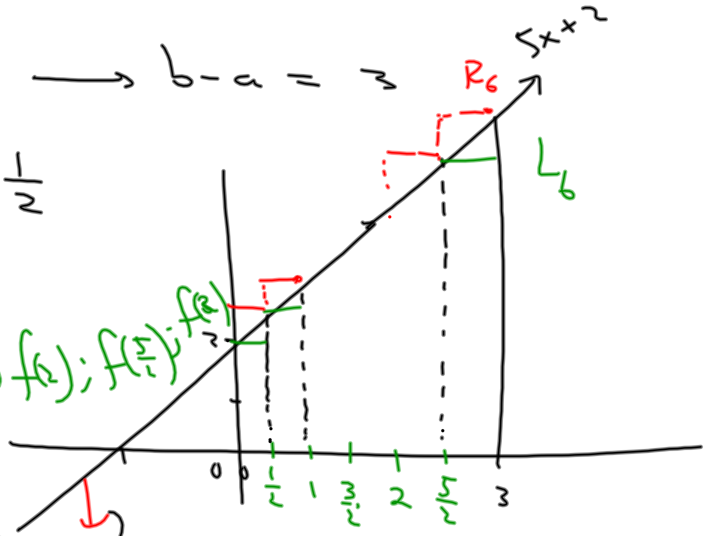
Ans:

$n=6$, $a=0$, $b=3$ $\longrightarrow$ $b-a=3$

base: $\dfrac{b-a}{n} = \dfrac{3}{6} = \dfrac{1}{2}$

heights:

$f(0)$; $f(\tfrac{1}{2})$; $f(1)$; $f(\tfrac{3}{2})$; $f(2)$; $f(\tfrac{5}{2})$; $f(3)$



$\{2, 4.5, 7, 9.5, 12, 14.5, 17\}$

L_6: $\frac{1}{2}[2+4.5+7+9.5+12+14.5] \approx \boxed{24.75}$

R_6: $\frac{1}{2}[4.5+7+9.5+12+14.5+17] \approx \boxed{32.25}$

Error of the approximation

The actual area: $A = \left(\dfrac{2+17}{2}\right)\underbrace{(3)}_{\text{base}}$

$\left(\dfrac{h_1+h_2}{2}\right)$ base = Area of a trapeze!

$A =$ Area: $\underline{28.5 \text{ unit}^2}$

Error: $A - L_6 \approx 3.75$

$R_6 - A \approx 3.75$

Remark:

$$\frac{R_N + L_N}{2} = \text{Trapezoidal Rule}$$

gives a better approximation as it cuts the "error" into half!

Here, we would have from the trapezoidal rule $A \simeq \frac{24.75 + 32.25}{2}$

$$\boxed{A \simeq 28.5 \text{ unit}^2}$$

- Sigma Sum: $\sum$

used to write a "long form" of summation in a compact form

$$a_1 + a_2 + \cdots + a_n = \sum_{k=1}^{n} a_k$$

So, for $L_N = \Delta x \sum_{i=0}^{n-1} f(x_i)$, $\Delta x = \frac{b-a}{n}$

$$R_N = \Delta x \sum_{i=1}^{n} f(x_i)$$